Additive induced-hereditary properties and unique factorization

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Abstract

We show that additive induced-hereditary properties of coloured hypergraphs can be uniquely factorised into irreducible factors. Our constructions and proofs are so general that they can be used for arbitrary concrete categories of combinatorial objects; we provide some examples of such combinatorial objects.

Keywords: coloured hypergraph, digraph, (induced-) hereditary property, combinatorial system, unique factorization, concrete category

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1 Introduction

Many problems treated in graph theory concern graph properties. Roughly speaking, a graph property is a subset of the set of all graphs, such as the family of planar graphs, perfect graphs, interval graphs, claw-free graphs or hamiltonian graphs. Some of these properties have important common features that allow us to study them from a more general point of view.

From a combinatorial aspect there is usually no need to distinguish isomorphic copies of graphs and we therefore restrict our attention to unlabeled graphs. More precisely, we require that a graph property be closed under isomorphism.

Many properties have the important feature of being also closed under taking some substructures. Consider a partial ordering $\preceq$ defined on the set of graphs. A property $\mathcal{P}$ is $\preceq$*-hereditary* if, whenever G belongs to $\mathcal{P}$ and $H \preceq G$, then H belongs to $\mathcal{P}$ as well. Greenwell et al. proved in [8] that such properties are exactly those that can be characterized by the set of $\preceq$-minimal forbidden substructures, assuming that there is no infinite descending $\preceq$-chain of structures. For example, the properties of k-degenerate graphs, claw-free graphs and planar graphs are, respectively, subgraph-hereditary, induced-subgraph-hereditary, and minor-hereditary.

Another important feature of many properties is that they are closed under disjoint union of graphs. Such properties are said to be *additive*. We show that this feature plays a substantial role in the study of the structure of $\preceq$-hereditary properties.

The language of graph properties can be successfully used to generalize ordinary vertex colouring. In a *proper colouring* each colour class must be an independent set. In so-called *generalized colouring*, each colour class must have a prescribed graph property. Given a list of two or more properties, the class of all graphs that can be coloured according to that list is said to be a *reducible* property. One can immediately ask whether different lists correspond to different properties. This is the *unique factorisation problem*, which was solved affirmatively for additive hereditary and additive induced-hereditary graph properties in [11, 10, 7].

In this paper we extend these results to induced-hereditary properties of directed coloured hypergraphs. Moreover we show that our result can be generalised beyond graphs and hypergraphs to other combinatorial objects such as oriented graphs, or partially ordered sets.

In Section 2 we introduce basic concepts and definitions that are used throughout the rest of the paper. In Section 3 we prove some necessary preliminary results. Section 4 is devoted to canonical factorisations of induced-hereditary properties of hypergraphs. The Unique Factorisation Theorem for hypergraphs is presented in Section 5. In the sixth section we introduce systems of objects of a concrete category, give some examples, and prove the Unique Factorisation Theorem for such systems.

2 Basic concepts and definitions

In general we use standard graph and hypergraph terminology that can be found, say, in [1, 2]. For terminology related to hereditary properties of graphs and hypergraphs we follow [3]. In the next three sections we restrict our attention to finite hypergraphs, without loops (hyperedges of size 1) or multiple hyperedges. For the sake of brevity, we sometimes drop the “hyper” prefix, using “edge” instead of “hyperedge”.

We will take our edges to be coloured and directed, so each edge is not a set but an ordered tuple $(v_1, \ldots, v_r; c)$, where the v_i's are the vertices of the edge, and c is its colour. Isomorphisms must preserve the colour and direction of each edge. The direction and colour actually make no difference, and are never mentioned in the proofs, so the reader might find it easier to think about hypergraphs without colours or directions. We also point out in advance that, if our properties contain only k-uniform hypergraphs, all our constructions will only give k-uniform hypergraphs. Similarly, we may restrict ourselves to hypergraphs with edge-colours taken from a prescribed set.

A *hypergraph property* is any non-empty isomorphism-closed subclass of hypergraphs. If H belongs to a property $\mathcal{P}$, then we also say that H has property $\mathcal{P}$. The subhypergraph of H induced by $U \subseteq V(H)$ is $H[U]$, with edge-set $E(H[U]) := \{e \in E(H) | e \subseteq U\}$. H' is an induced-subhypergraph of H if it is isomorphic to $H[U]$ for some $U \subseteq H$, and we write $H' \leq H$.

A property $\mathcal{P}$ is *induced-hereditary* if $H \in \mathcal{P}$ implies that $K \in \mathcal{P}$, for all $K \leq H$. A property is *additive* if it is closed under taking disjoint union of hypergraphs. More precisely, if $H_1 = (V_{H_1}, E_{H_1})$, $H_2 = (V_{H_2}, E_{H_2})$ are hypergraphs with $V(H_1) \cap V(H_2) = \emptyset$, then their disjoint union is the hypergraph $K = (V_{H_1} \cup V_{H_2}, E_{H_1} \cup E_{H_2})$. A hypergraph is connected if and only if it cannot be expressed as a disjoint union of two hypergraphs.

Following the arguments in [4] and [9] one can easily verify that the set of all induced-hereditary properties of hypergraphs ordered by set inclusion forms a completely distributive algebraic lattice, which we shall denote by $\mathbb{H}^a_{\leq}$. For many more details, applications and open problems concerning hereditary and induced-hereditary properties we refer the reader to [3].

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of hypergraphs. A $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partition* of a hypergraph H is a partition $(V_1, V_2, \ldots, V_n)$ of the vertex set $V(H)$ such that the induced subhypergraph $H[V_i]$ has property $\mathcal{P}_i$, for $i = 1, 2, \ldots, n$. Note that V_i could be empty for any i; equivalently, one can assume the null graph $K_0 = (\emptyset, \emptyset)$ to be contained in every property. If a hypergraph H has a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition, then we say that H has property $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. If $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n$ we simply write $\mathcal{P}^n$ instead of $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$.

Let $\mathcal{P}$ be additive induced-hereditary; $\mathcal{P}$ is *reducible* if there are additive induced-hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$; otherwise, it is *irreducible*. One may consider an alternative definition of reducibility in which $\mathcal{P}_1$ and $\mathcal{P}_2$ can be any two properties, not necessarily additive induced-hereditary. The two definitions turn out to be equivalent, but the proof of this non-trivial fact depends on the Unique Factorisation Theorem, and a further result characterising the existence of uniquely colourable graphs [5, 6], so we will stick with the first definition.

Unless stated otherwise, the properties we consider are additive induced-hereditary hypergraph properties. We will consider more general properties in the last section.

3 Uniquely decomposable hypergraphs

The main result of this section is the existence of uniquely $\mathcal{P}$-decomposable hypergraphs, for every additive induced-hereditary property $\mathcal{P}$. In fact, every hypergraph in $\mathcal{P}$ is an induced-subhypergraph of a uniquely $\mathcal{P}$-decomposable hypergraph.

Let $\mathcal{G}$ be a set of hypergraphs. The induced-hereditary property *generated by* $\mathcal{G}$ is $\langle \mathcal{G} \rangle$, the

smallest induced-hereditary property containing $\mathcal{G}$. $\mathcal{G}$ is a *generating set* for $\mathcal{P}$ if $\langle \mathcal{G} \rangle = \mathcal{P}$. It is easy to see that:

$$\langle \mathcal{G} \rangle \quad = \quad \{G \mid \exists\, H \in \mathcal{G},\ G \leq H\}.$$

The $*$-*join* of n hypergraphs $G_1, \ldots, G_n$ with disjoint vertex-sets is the set of all hypergraphs obtained by adding edges between the G_i's; no new edges $e \subseteq V(G_i)$ are added:

$$G_1 * \cdots * G_n := \{H \mid V(H) = \bigcup_{i=1}^{n} V(G_i), H[V(G_i)] = G_i\}.$$

Given n sets of hypergraphs, we define their $*$-join by

$$S_1 * \cdots * S_n := \bigcup (G_1 * \cdots * G_n)$$

the union being over all ways of the selecting the G_i's so that $G_i \in S_i$ for all i. We note that this is just the same as $S_1 \circ \cdots \circ S_n$, but it is aesthetically pleasing to have the $*$ notation.

If $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are additive properties, and $G_i \in \mathcal{P}_i$ for all i, then for all positive integers k we have

$$kG_1 * \cdots * kG_n \subseteq \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$$

where kG is the disjoint union of k copies of G. A *$\mathcal{P}$-decomposition of G with n parts* is a partition $(V_1, \ldots, V_n)$ of $V(G)$ such that for all i, $V_i \neq \emptyset$, and for all positive integers k we have $kG[V_1] * \cdots * kG[V_n] \subseteq \mathcal{P}$. The *$\mathcal{P}$-decomposability number $dec_{\mathcal{P}}(G)$* of G is the maximum number of parts in a $\mathcal{P}$-decomposition of G; for $G \notin \mathcal{P}$ we put $dec_{\mathcal{P}}(G) = 0$. Thus G is in $\mathcal{P}$ if and only if $dec_{\mathcal{P}}(G) \geq 1$. Also, G is *$\mathcal{P}$-decomposable* if $dec_{\mathcal{P}}(G) > 1$. If $\mathcal{P}$ is the product of two additive induced-hereditary properties, then *every* hypergraph in $\mathcal{P}$ with at least two vertices is $\mathcal{P}$-decomposable.

Lemma 3.1 *Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$, where the $\mathcal{P}_i$'s are additive properties. Then any $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition of a hypergraph G is a $\mathcal{P}$-decomposition of G. If the $\mathcal{P}_i$'s are induced-hereditary, then every hypergraph in $\mathcal{P}$ with at least m vertices has a partition with all m parts non-empty.* $\square$

A hypergraph G is *$\mathcal{P}$-strict* if G is in $\mathcal{P}$ but $G * K_1 \not\subseteq \mathcal{P}$; we denote the set of $\mathcal{P}$-strict hypergraphs by $\mathbf{S}(\mathcal{P})$. If $f(\mathcal{P}) = \min\{|V(F)| \mid F \notin \mathcal{P}\}$, then $G * K_1 * \cdots * K_1 \not\subseteq \mathcal{P}$, where the $*$ operation is repeated $f(\mathcal{P})$ times. Thus, every $G \in \mathcal{P}$ is an induced-subhypergraph of some $\mathcal{P}$-strict hypergraph (with fewer than $|V(G)| + f(\mathcal{P})$ vertices), and so $\langle \mathbf{S}(\mathcal{P}) \rangle = \mathcal{P}$. Similarly, $dec_{\mathcal{P}}(G) < f(\mathcal{P})$.

The *$\mathcal{P}$-decomposability number $dec_{\mathcal{P}}(\mathcal{G})$* of a generating set $\mathcal{G}$ of $\mathcal{P}$ is

$$\min\{dec_{\mathcal{P}}(G) \mid G \in \mathcal{G}\};$$

the *decomposability number $dec(\mathcal{P})$* of $\mathcal{P}$ is $dec_{\mathcal{P}}(\mathbf{S}(\mathcal{P}))$. A property with $dec(\mathcal{P}) = 1$ is *indecomposable*. An indecomposable property is also irreducible and it will turn out that the converse is also true.

Lemma 3.2 *Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be induced-hereditary properties, and let G be a $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$-strict hypergraph. Then, for every $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition $(V_1, \ldots, V_m)$ of $V(G)$, $G[V_i]$ is $\mathcal{P}_i$-strict (and in particular non-empty).*

Proof. If $G[V_1] * K_1 \subseteq \mathcal{P}_1$, then $G * K_1 \subseteq (G[V_1] * K_1) * G[V_2] * \cdots * G[V_m] \subseteq \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$. $\square$

It follows that $dec(\mathcal{A} \circ \mathcal{B}) \geq dec(\mathcal{A}) + dec(\mathcal{B})$, and thus any factorisation of an additive induced-hereditary property $\mathcal{P}$ has at most $dec(\mathcal{P})$ irreducible additive induced-hereditary factors.

Lemma 3.3 *[11] Let $\mathcal{P}$ be an induced-hereditary property and G be a $\mathcal{P}$-strict induced subhypergraph of $G' \in \mathcal{P}$. Then G' is $\mathcal{P}$-strict, and $dec_{\mathcal{P}}(G) \geq dec_{\mathcal{P}}(G')$.*

Proof. Every hypergraph in $G * K_1$ is an induced subhypergraph of a hypergraph in $G' * K_1$, so G' must be $\mathcal{P}$-strict. If $(V_1, \ldots, V_n)$ is a $\mathcal{P}$-decomposition of G' with n parts, then $(V_1 \cap V(G), \ldots, V_n \cap V(G))$ is a $\mathcal{P}$-decomposition of G; moreover, it has n parts unless, for some i, $V_i \cap V(G) = \emptyset$, which is impossible because G is $\mathcal{P}$-strict. $\square$

Lemma 3.4 *[11] If $\mathcal{G}$ generates the induced-hereditary property $\mathcal{P}$, then $dec_{\mathcal{P}}(\mathcal{G}) \leq dec_{\mathcal{P}}(\mathbf{S}(\mathcal{P}))$, with equality if $\mathcal{G} \subseteq \mathbf{S}(\mathcal{P})$.* $\square$

For $\mathcal{G} \subseteq \mathcal{P}$, and $H \in \mathcal{P}$, let $\mathcal{G}[H] := \{G \in \mathcal{G} \mid H \leq G\}$.

Lemma 3.5 *[11] Let $\mathcal{G}$ generate the additive induced-hereditary property $\mathcal{P}$, and let H be an arbitrary hypergraph in $\mathcal{P}$. Then $\mathcal{G}[H]$ also generates $\mathcal{P}$.* $\square$

For a generating set $\mathcal{G}$, let $\mathcal{G}^{\downarrow} := \{G \in \mathcal{G} \mid G \in \mathbf{S}(\mathcal{P}),\ dec_{\mathcal{P}}(G) = dec(\mathcal{P})\}$. The following is a simple consequence of Lemmas 3.3 and 3.5.

Lemma 3.6 *[11] If $\mathcal{G}$ generates the additive induced-hereditary property $\mathcal{P}$, then so does $\mathcal{G}^{\downarrow}$.* $\square$

A hypergraph G is *uniquely $\mathcal{P}$-decomposable* if it has exactly one $\mathcal{P}$-decomposition with $dec_{\mathcal{P}}(G)$ parts. Equivalently, G is either $\mathcal{P}$-indecomposable, or has exactly one $\mathcal{P}$-decomposition with n parts, for some $n \geq 2$; in the second case, n must be $dec_{\mathcal{P}}(G)$, as any decomposition with $n+1$ parts would give rise to $\binom{n+1}{2}$ decompositions with n parts.

If $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$, then by Lemma 3.1 a uniquely $\mathcal{P}$-decomposable hypergraph G with $dec_{\mathcal{P}}(G) = n$ must be uniquely $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$-partitionable (every $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$-partition gives the same unordered partition of $V(G)$). If $(V_1, \ldots, V_n)$ is the unique $\mathcal{P}$-decomposition of G, we call the hypergraphs $G[V_1], \ldots, G[V_n]$ its *ind-parts* (although they are themselves usually $\mathcal{P}$-decomposable).

Lemma 3.7 *Let $\mathcal{P}$ be an induced-hereditary property and let G be a hypergraph in $\mathbf{S}(\mathcal{P})$ with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and suppose that G has a unique $\mathcal{P}$-decomposition $(V_1, \ldots, V_{dec(\mathcal{P})})$ with $dec(\mathcal{P})$ parts. If $G \leq H$, then $H \in \mathbf{S}(\mathcal{P})$, $dec_{\mathcal{P}}(H) = dec(\mathcal{P})$, and, for any $\mathcal{P}$-decomposition $(W_1, \ldots, W_{dec(\mathcal{P})})$ of H, we can relabel the W_i's so that, for all i, $W_i \cap V(G) = V_i$.* $\square$

Let $d_0 = (U_1, U_2, \ldots, U_m)$ be a $\mathcal{P}$-decomposition of a hypergraph G. A $\mathcal{P}$-decomposition $d_1 = (V_1, V_2, \ldots, V_n)$ of G *respects* d_0 if no V_i intersects two or more U_j's; that is, each V_i is contained in some U_j, and so each U_j is a union of V_i's.

If G is a hypergraph, then $s \oslash G$ denotes the set $G * G * \ldots * G$, where there are s copies of G. For $G^* \in s \oslash G$, denote the copies of G by $G^1, \ldots, G^s$. Then G^* *respects* d_0 if $G^* \in sG[U_1] * \cdots * sG[U_m]$; that is, an edge that intersects different G^i's must also intersect different U_j's. A $\mathcal{P}$-decomposition $d = (V_1, \ldots, V_n)$ of G^* *respects* d_0 *uniformly* if, for each V_i, there is a U_j such that, for every G^k, $V_i \cap V(G^k) \subseteq U_j$. The decomposition of G^k induced by d is denoted $d|G^k$.

If G is uniquely $\mathcal{P}$-decomposable, its ind-parts *respect* d_0 if its unique $\mathcal{P}$-decomposition with $dec_{\mathcal{P}}(G)$ parts respects d_0. If G^* is uniquely $\mathcal{P}$-decomposable, its ind-parts *respect* d_0 *uniformly* if: (a) for some s, $G^* \in s \oslash G$; (b) G^* respects d_0; and (c) G^*'s unique $\mathcal{P}$-decomposition with $dec_{\mathcal{P}}(G^*)$ parts respects d_0 uniformly.

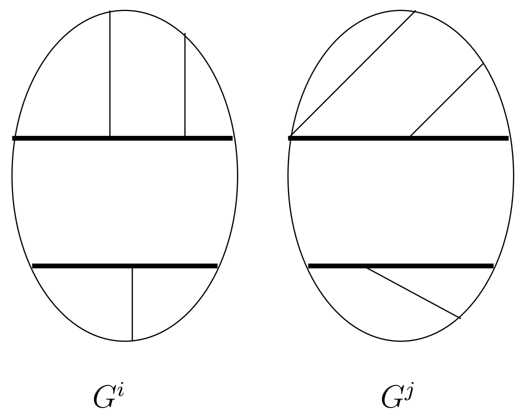

Figure 1: d (vertical lines) respects d_0 (horizontal lines) uniformly

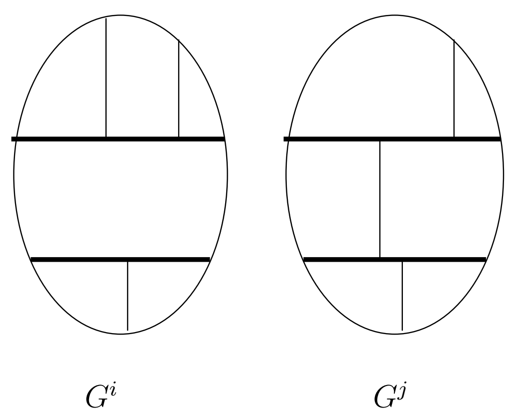

Figure 2: d (vertical lines) respects d_0 (horizontal lines) on both G^i and G^j, but not uniformly

The *extension of* d_0 to G^* is the decomposition obtained by repeating d_0 on each copy of G. If G^* respects d_0, or if it has a $\mathcal{P}$-decomposition that respects d_0 uniformly, then the extension of d_0 is also a $\mathcal{P}$-decomposition of G^*. In particular, G^* is a hypergraph in $\mathcal{P}$.

We will sometimes write $G^i \cap U_x$ (or just U_x when it is clear we are referring to G^i) to mean the vertices of G^i that correspond to U_x, and $G^* \cap U_x$ (or just U_x, when it is clear from the context) to mean $G^*[\bigcup_i (G^i \cap U_x)]$.

The required result is a corollary of the following theorem of Mihók (see [10]); he actually proved it when $m = n$ (Corollary 3.10), but very little modification is needed to establish the general case, and we follow his proof and notation rather closely.

Theorem 3.8 *Let G be a $\mathcal{P}$-strict hypergraph with $dec_{\mathcal{P}}(G) = n$, and let $d_0 = (U_1, U_2, \ldots, U_m)$ be a fixed $\mathcal{P}$-decomposition of G. Then there is a $\mathcal{P}$-strict hypergraph $G^* \in s \oslash G$ (for some s) that respects d_0, and moreover any $\mathcal{P}$-decomposition of G^* with n parts respects d_0 uniformly.*

Proof. Let $d_i = (V_{i,1}, V_{i,2}, \ldots, V_{i,n})$, $i = 1, \ldots, r$, be the $\mathcal{P}$-decompositions of G with n parts which do not respect d_0. Since G is a finite hypergraph, r is a nonnegative integer. If $r = 0$, take $G^* = G$; otherwise we will construct a hypergraph $G^* = G^*(r) \in s \oslash G$ as above, denoting the s copies of G by $G^1, \ldots, G^s$.

If the resulting G^* has a $\mathcal{P}$-decomposition d with n parts, then, since G is $\mathcal{P}$-strict, $d|G^i$ will also have n parts. The aim of the construction is to add new edges $E^* = E^*(r)$ to sG to exclude the possibility that $d|G^i = d_j$, for any $1 \le i \le s, 1 \le j \le r$. Whenever we add an edge e, if e intersects $G^i \cap U_x$, it will also intersect some $G^j, i \neq j$, and some $U_y, x \neq y$; thus G^* will respect d_0, and the hypergraphs constructed will always be in $\mathcal{P}$.

We shall use two types of constructions.

Construction 1. $G^i \Rightarrow G^j$.

This is a hypergraph in $2 \oslash G$ such that, if d is a $\mathcal{P}$-decomposition of $G^i \Rightarrow G^j$ and $d|G^i$ respects d_0, then $d|G^j$ respects d_0; moreover, d respects d_0 uniformly on $G^i \Rightarrow G^j$.

Since G is $\mathcal{P}$-strict, there is a hypergraph $F \in (G * K_1) \setminus \mathcal{P}$. Let E' be the edges of F that contain $z \in V(K_1)$. For $x = 1, 2, \ldots, m$, let E'_x be the set of edges from E' that contain only z and vertices of U_x, while $E'_{\overline{x}}$ is the set of edges from E' that contain some vertex of $V(G) \setminus U_x$. Let $G^i, G^j, i \neq j$, be disjoint copies of G; for every x, and every vertex $v \in U_x \cap V(G_j)$, we add the edges of $E'_{\overline{x}}$ (with v taking the place of z, and G_i taking the place of G). Note that $G^i \Rightarrow G^j \in 2G[U_1] * 2G[U_2] * \cdots * 2G[U_m]$. Since d_0 is a $\mathcal{P}$-decomposition of G, this implies that $(G^i \Rightarrow G^j) \in \mathcal{P}$.

Let $d = (V_1, V_2, \ldots, V_\ell)$ be a $\mathcal{P}$-decomposition of $H = (G^i \Rightarrow G^j)$ such that $d|G^i$ respects d_0, but $d|G^j$ does not respect d_0 (or at least, not in the same manner, i.e., d does not respect d_0 uniformly). Then there exist k and $x \neq y$ such that $V_k \cap G^i \subseteq U_y$, but some $v \in V_k \cap G^j$ belongs to U_x. We can add the edges corresponding to $E' \setminus E'_x$ because they contain at least one vertex w of $G^i \cap U_x$ (so $w \notin V_k$). But then, F is an induced subhypergraph of a hypergraph in $H[V_1] * H[V_2] * \ldots * H[V_\ell]$, which implies $F \in \mathcal{P}$, a contradiction.

Construction 2. $m \bullet k_t G$.

For a $\mathcal{P}$-decomposition $d_t = (V_{t,1}, V_{t,2}, \ldots, V_{t,dec_{\mathcal{P}}(G)})$ of G that does not respect d_0, $m \bullet k_t G$ is a hypergraph in $(mk_t) \oslash G$ having no $\mathcal{P}$-decomposition $d = (W_1, W_2, \ldots, W_{dec_{\mathcal{P}}(G)})$ such that, for all of the mk_t induced copies G^i of G, $d|G^i = d_t$.

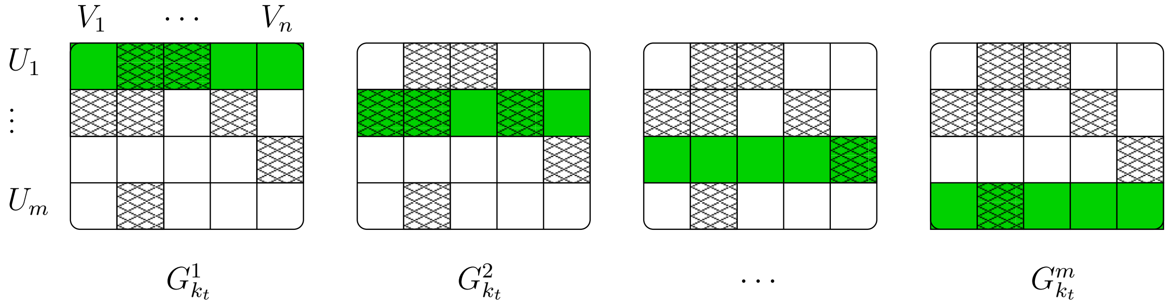

Figure 3: $m \bullet k_tG$ - we only put edges between the m shaded parts

Let $n = dec_{\mathcal{P}}(G)$ and let $A_{i,j}(t)$ denote $U_i \cap V_{t,j}, 1 \leq i \leq m, 1 \leq j \leq n$. Since d_t does not respect d_0, at least $n+1$ sets $A_{i,j}(t)$ are nonempty. Because $dec_{\mathcal{P}}(G) = n$, there exists a positive integer k_t such that $k_tG[A_{1,1}(t)] * k_tG[A_{1,2}(t)] * \ldots * k_tG[A_{m,n}(t)] \not\subset \mathcal{P}$. Fix a hypergraph $F_t \in (k_tG[A_{1,1}(t)] * k_tG[A_{1,2}(t)] * \ldots * k_tG[A_{m,n}(t)]) \setminus \mathcal{P}$. Note that F_t differs from k_tG only in the edges that intersect at least two different U_i's, or at least two different V_j's.

The U_i's form a $\mathcal{P}$-decomposition of k_tG, so we can replace the edges of k_tG that intersect at least two U_i's, with the edges of F_t that intersect at least two U_i's, and still remain in $\mathcal{P}$. If, in the resulting hypergraph $\tilde{H}$, the V_j's also formed a $\mathcal{P}$-decomposition, we could replace the edges of $\tilde{H}$ that intersect at least two different V_j's with the edges of F_t that intersect at least two different V_j's, and still remain in $\mathcal{P}$. But this is impossible because we would then have $F_t \in \mathcal{P}$.

The only problem with $\tilde{H}$ is that, in order to construct it, we altered edges *inside* the k_t copies that we had of G. We therefore construct $m \bullet k_tG$ by taking m disjoint copies of $H = k_tG$, denoted by $H^j, j = 1, 2, \ldots m$, and adding edges between $H^1 \cap U_1, H^2 \cap U_2, \ldots, H^m \cap U_m$. Specifically, suppose an edge of F_t intersects $U_{a_1}, \ldots, U_{a_r}$ $(1 \leq a_1 < \cdots < a_r \leq m,\ r \geq 2)$; then in $m \bullet k_tG$ we put a corresponding edge that intersects $H^{a_1} \cap U_{a_1}, \ldots, H^{a_r} \cap U_{a_r}$.

Suppose $d = (W_1, W_2, \ldots, W_{dec_{\mathcal{P}}(G)})$ is a $\mathcal{P}$-decomposition of $m \bullet k_tG$ such that, for every one of the mk_t induced copies G^i of G, $d|G^i = d_t$. Then $H^1 \cap U_1, \ldots, H^m \cap U_m$ induce a copy of the hypergraph $\tilde{H}$ from which we could obtain F_t while still remaining in $\mathcal{P}$, thus getting a contradiction as above.

We now construct G^* as follows. First let $G(0) := G$ and $G(1) := m \bullet k_1G$. For $1 < \ell \leq r$, construct $G(\ell)$ by taking mk_ℓ disjoint copies $G(\ell-1)^1, \ldots, G(\ell-1)^{mk_\ell}$ of $G(\ell-1)$. For each copy of G in $G(\ell-1)^i$ and each copy of G in $G(\ell-1)^j$, we add the edges between them that are between the i^{th} and j^{th} copies of G in $m \bullet k_\ell G$. (See Figure 4.)

Finally, from $G(r)$, which is in, say, $s \oslash G$, consisting of copies G^1, G^2, $\ldots$, G^s of G, we create G^* by adding two more copies G^+ and G^- of G. We add edges between G^+ and G^- to create the hypergraph $G^- \Rightarrow G^+$, and, for each $i = 1, \ldots, s$, we add edges to obtain $G^i \Rightarrow G^-$ and $G^+ \Rightarrow G^i$.

Let d be a $\mathcal{P}$-decomposition of G^* with n parts (it might be that none exists, in which case we are done). For $1 \leq \ell \leq r$, if every copy of $G(\ell-1)$ in $G(\ell)$ contains a copy of G for which $d|G = d_\ell$, then we would have mk_ℓ such copies of G inducing a copy of $m \bullet k_\ell G$, which we know is impossible. So by induction from r to 1, there is a copy G^p of G for which $d|G^p$ is none of $d_1, d_2, \ldots, d_r$. Thus, $d|G^p$ respects d_0. But $G^p \Rightarrow G^-$ is an induced subgraph of G^*,

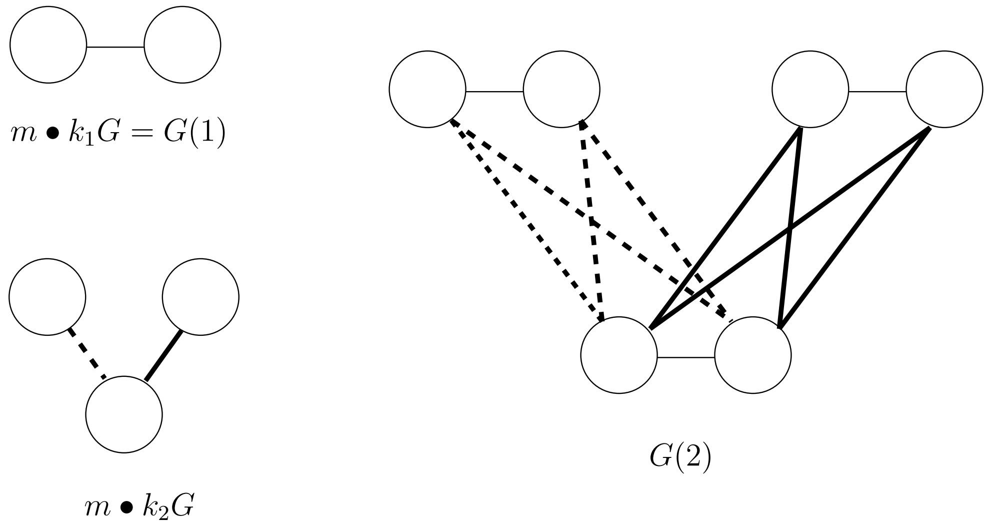

Figure 4: Constructing $G(2)$ from $G(1)$ and $m \bullet k_2 G$

so $d|G^- = d_0$ (and in fact d respects d_0 uniformly on these two copies of G). Similarly, $d|G^+$ respects d_0 and, in the same way, d respects d_0 uniformly, as required. □

Corollary 3.9 *Let G be a $\mathcal{P}$-strict hypergraph with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and let $d_0 = (U_1, U_2, \ldots, U_m)$ be a fixed $\mathcal{P}$-decomposition of G. Then there is a $\mathcal{P}$-decomposition of G with exactly $dec(\mathcal{P})$ parts that respects d_0.*

Proof. In Theorem 3.8, since $G^* \geq G$ we know G^* is $\mathcal{P}$-strict, and so $dec(\mathcal{P}) \leq dec_{\mathcal{P}}(G^*) \leq dec_{\mathcal{P}}(G) = dec(\mathcal{P})$. Thus G^* has at least one $\mathcal{P}$-decomposition d with $dec(\mathcal{P})$ parts; $d|G$ also has $dec(\mathcal{P})$ parts (since G is $\mathcal{P}$-strict) and respects d_0. □

Corollary 3.10 *[11] Let G be a $\mathcal{P}$-strict hypergraph with $dec_{\mathcal{P}}(G) = n$, and let $d_0 = (U_1, U_2, \ldots, U_n)$ be a fixed $\mathcal{P}$-decomposition of G with n parts. Then there is a $\mathcal{P}$-strict hypergraph $G^* \in s \oslash G$ (for some s) which has a unique $\mathcal{P}$-decomposition d with n parts, and $d|G^j = d_0$ for all j.*

Proof. The only $\mathcal{P}$-decomposition of G with n parts that respects d_0 is d_0 itself (since here d_0 has exactly n parts). Thus in Theorem 3.8, the only possible decomposition of G^* with n parts is the extension of d_0, which is a $\mathcal{P}$-decomposition of G^* by construction. □

The set of $\mathcal{P}$-strict, uniquely $\mathcal{P}$-decomposable hypergraphs with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$ is denoted $\mathbf{S}^{\Downarrow}(\mathcal{P})$, or just $\mathbf{S}^{\Downarrow}$. By Lemma 3.6 and Corollary 3.10 $\mathbf{S}^{\Downarrow}$ is a generating set for $\mathcal{P}$; in fact, for any $G \in \mathbf{S}^{\Downarrow}$ and any specific $\mathcal{P}$-decomposition d of G, we can find a hypergraph in $\mathbf{S}^{\Downarrow}$ that contains G and whose ind-parts uniformly respect d.

Corollary 3.11 *Let G be a $\mathcal{P}$-strict hypergraph with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and let $d_0 = (U_1, U_2, \ldots, U_m)$ be a fixed $\mathcal{P}$-decomposition of G. Then there is a uniquely $\mathcal{P}$-decomposable $\mathcal{P}$-strict hypergraph $G^* \geq G$ whose ind-parts respect d_0 uniformly.* □

4 Canonical factorisations

A property $\mathcal{P}$ is *indecomposable* if $dec(\mathcal{P}) = 1$. In this section we show that every additive induced-hereditary property $\mathcal{P}$ has a factorisation into $dec(\mathcal{P})$ additive induced-hereditary properties; this establishes the important fact that $\mathcal{P}$ is irreducible iff it is indecomposable. In the next section we will show that whenever $\mathcal{P}$ has a factorisation into indecomposable factors, there must be exactly $dec(\mathcal{P})$ of them, and this factorisation must be unique.

Lemma 4.1 *[13] A generating set $\mathcal{G}' = \{G_1, G_2, \ldots\}$ for $\mathcal{P}$ contains an ordered generating set $\mathcal{G}^{\odot} = \{G_{k_1}, G_{k_2}, \ldots\} \subseteq \mathcal{G}'$ satisfying $G_{k_1} \leq G_{k_2} \leq \cdots$.*

Proof. Since $\mathcal{P}$ contains only finite graphs, it is countable, say $\mathcal{P} = \{H_1, H_2, \ldots\}$. Pick G_{k_1} arbitrarily. For each i, by additivity, $G_{k_i} \cup H_i$ is in $\mathcal{P}$, so there is a k_{i+1} for which $(G_{k_i} \cup H_i) \leq G_{k_{i+1}}$. $\square$

Recall that $\mathbf{S}^{\Downarrow} := \mathbf{S}^{\Downarrow}(\mathcal{P})$ is the set of uniquely $\mathcal{P}$-decomposable hypergraphs with decomposability $n := dec(\mathcal{P})$. By Lemma 3.6 and Corollary 3.10, $\mathbf{S}^{\Downarrow}$ is a generating set for $\mathcal{P}$. By Lemma 4.1 there is an ordered generating set $\mathcal{G} \subseteq \mathbf{S}^{\Downarrow}$ for $\mathcal{P}$.

For a hypergraph $G \in \mathbf{S}^{\Downarrow}$ whose unique $\mathcal{P}$-decomposition is $(V_1, \ldots, V_n)$, the set of ind-parts is $Ip\ (G) := \{G[V_1], \ldots, G[V_n]\}$. The set of all ind-parts from $\mathcal{G}$ is $I_{\mathcal{G}} := \bigcup(Ip\ (G) : G \in \mathcal{G})$. For $F \in I_{\mathcal{G}}$ and $G \in \mathcal{G}$, $m(F, G)$ is the *multiplicity* of F in G: the number of different (possibly isomorphic) ind-parts of G which contain F as an induced-subhypergraph. The multiplicity of F in $\mathcal{G}$ is $m(F) = \max\{m(F, G) \mid G \in \mathcal{G}\}$; clearly $1 \leq m(F) \leq n = dec(\mathcal{P})$.

Lemma 4.2 *Let G, H, be hypergraphs in $\mathbf{S}^{\Downarrow}$. If $G \leq H$, then each ind-part of G is an induced-subhypergraph of a distinct ind-part of H.*

Proof. Let the ind-parts of G and H be $(G_1, G_2, \ldots G_n)$ and $(H_1, \ldots, H_n)$, respectively. $(H_1 \cap G, H_2 \cap G, \ldots, H_n \cap G)$ is a $\mathcal{P}$-decomposition of G, where we use $H_k \cap G$ to denote $V(H_k) \cap V(G)$. If $H_k \cap G = \emptyset$ for some k, then G would not be $\mathcal{P}$-strict, a contradiction. So we have a $\mathcal{P}$-decomposition of G with n parts. Because G is in $\mathcal{G}$, there is only one such decomposition, so without loss of generality $H_k \cap G = V(G_k),\ k = 1, \ldots, n$. $\square$

For convenience, we will talk of the hypergraph induced by ind-parts $G_1, G_2, \ldots$, when we actually mean the subhypergraph induced by $V(G_1) \cup V(G_2) \cup \cdots$.

Theorem 4.3 *[11] An additive induced-hereditary property $\mathcal{P}$ has a factorisation into $dec(\mathcal{P})$ (necessarily indecomposable) additive induced-hereditary factors.*

Proof. We proceed by induction on $dec(\mathcal{P})$. If $dec(\mathcal{P}) = 1$ there is nothing to do. So let every hypergraph $G \in \mathbf{S}(\mathcal{P})$ with at least two vertices be $\mathcal{P}$-decomposable. We will either factorise at once into $n := dec(\mathcal{P})$ properties, or into properties $\mathcal{Q}, \mathcal{R}$ such that $dec(\mathcal{P}) = dec(\mathcal{Q}) + dec(\mathcal{R})$.

Case 1. $m(F) = k < dec(\mathcal{P})$, for some $F \in I_{\mathcal{G}}$.

Let $G \in \mathcal{G}$ be a generator of $\mathcal{P}$ for which $m(F, G) = k$. By Lemma 3.5, $\mathcal{G}[G]$ generates $\mathcal{P}$; by Lemma 4.2, for every generator $H \in \mathcal{G}[G]$, $m(F, H) = k$, so $\mathcal{G}_F := \{G' \in \mathcal{G} \mid m(F, G') = k\}$ is a generating set. For $H \in \mathcal{G}_F$, let H_F be the subgraph induced by the k ind-parts which contain

F, and $H_{\overline{F}}$ the subhypergraph induced by the $n-k$ other ind-parts. Let the induced-hereditary properties $\mathcal{Q}_F$ and $\mathcal{Q}_{\overline{F}}$ be generated by $\{H_F \mid H \in \mathcal{G}_F\}$ and $\{H_{\overline{F}} \mid H \in \mathcal{G}_F\}$, respectively.

We claim that $\mathcal{P} = \mathcal{Q}_F \circ \mathcal{Q}_{\overline{F}}$. It is easy to see that $\mathcal{P} \subseteq \mathcal{Q}_F \circ \mathcal{Q}_{\overline{F}}$. Conversely, let H be in $\mathcal{Q}_F \circ \mathcal{Q}_{\overline{F}}$. Then $H \in H^1_F * H^2_{\overline{F}}$, for some $H^1, H^2 \in \mathcal{G}_F$. Let H' be a hypergraph in $\mathcal{G}$ such that $H^1 \cup H^2 \leq H'$. By Lemma 4.2, and because the maximum multiplicity of F in $\mathcal{G}$ is k, $H^1_F \leq H'_F$ and $H^2_{\overline{F}} \leq H'_{\overline{F}}$. Since $H'_F * H'_{\overline{F}} \subseteq \mathcal{P}$, we have $H^1_F * H^2_{\overline{F}} \subseteq \mathcal{P}$, implying $H \in \mathcal{P}$. Hence $\mathcal{P} = \mathcal{Q}_F \circ \mathcal{Q}_{\overline{F}}$.

To establish additivity of $\mathcal{Q}_F$, consider $G_F, H_F \in \mathcal{Q}_F$, for some $G, H \in \mathcal{G}_F$. Because $\mathcal{G}_F$ generates $\mathcal{P}$, there is some $L \in \mathcal{G}_F$ such that $(G \cup H) \leq L$. By Lemma 4.2, $G_F \leq L_F$ and $H_F \leq L_F$, so $(G_F \cup H_F) \leq L_F \in \mathcal{Q}_F$, and thus $(G_F \cup H_F) \in \mathcal{Q}_F$. Additivity of $\mathcal{Q}_{\overline{F}}$ is proved similarly.

Finally, we want to show that every $H_F \in \mathcal{G}_F$ has $\mathcal{Q}_F$-decomposability at least k, and every $H_{\overline{F}} \in \mathcal{G}_{\overline{F}}$ has $\mathcal{Q}_{\overline{F}}$-decomposability at least $n-k$. This will imply that $n = dec(\mathcal{P}) \geq dec(\mathcal{Q}_F) + dec(\mathcal{Q}_{\overline{F}}) \geq k + (n-k) = n$, and thus $dec(\mathcal{Q}_F) = k$ and $dec(\mathcal{Q}_{\overline{F}}) = n-k$. Since $k < n$ and $n-k < n$, the factorisation result will follow by induction.

So consider $H_F \in \mathcal{G}_F$, and let H'_F be in $H_1 * \cdots * H_k$, where $H_1, \ldots, H_k$ are the ind-parts of H that contain F. Consider $H'' := H \cup H'_F$; since $H \leq H''$, H'' is $\mathcal{P}$-strict, and $dec_{\mathcal{P}}(H'') = dec(\mathcal{P})$; moreover, H'' has a $\mathcal{P}$-decomposition where the parts are $2H_1, \ldots, 2H_k, H_{k+1}, \ldots, H_n$. By Corollary 3.10, there is a uniquely $\mathcal{P}$-decomposable graph $H^* \in s\oslash H''$ whose ind-parts are $2sH_1, \ldots, 2sH_k, sH_{k+1}, \ldots, sH_n$. Thus $H'_F \leq H^*_F$, and so $H'_F \in \mathcal{Q}_F$. Since H'_F was arbitrary, $H_1, \ldots, H_k$ give a $\mathcal{Q}_F$-decomposition of H_F, and so $dec_{\mathcal{Q}_F}(H_F) \geq k$ as required. The proof that $dec(\mathcal{Q}_{\overline{F}}) \geq n-k$ is similar.

Case 2. $m(F) = n := dec(\mathcal{P}) \geq 2$ for each $F \in I_{\mathcal{G}}$.

Let $\mathcal{Q}$ be the induced-hereditary property generated by $I_{\mathcal{G}}$. It is easy to see that $\mathcal{P} \subseteq \mathcal{Q}^n$. The converse inclusion, $\mathcal{Q}^n \subseteq \mathcal{P}$, and the additivity and indecomposability of $\mathcal{Q}$, follow as in Case 1. □

Corollary 4.4 *[11] An additive induced-hereditary property is irreducible if and only if it is indecomposable.* □

5 Unique factorization theorem for hypergraphs

To prove unique factorisation, we shall first show that the number of factors must be exactly $dec(\mathcal{P})$, and then show that any two factorisations with $dec(\mathcal{P})$ factors must be the same. In the case of additive hereditary properties, there is a simple direct proof of the following result implicit in [10, Lemma 2.1].

Lemma 5.1 *Let $\mathcal{P}$ be $\mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_m$. Let G be a $\mathcal{P}$-strict, uniquely $\mathcal{P}$-decomposable graph with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and let $(W_1, \ldots, W_m)$ be a $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition of G. Then each W_j is a union of ind-parts of G.*

Proof. By Corollary 3.11 there is a a uniquely $\mathcal{P}$-decomposable graph $G^* \geq G$ whose ind-parts respect the W_j's. Now the ind-parts of G are just the restriction of the ind-parts of G^*. □

Theorem 5.2 *Let $\mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_m$ be a factorisation of the additive induced-hereditary property $\mathcal{P}$ into indecomposable additive induced-hereditary properties. Then $m = dec(\mathcal{P})$.*

Proof. By Lemma 3.1 any $\mathcal{P}$-strict graph G has $dec_{\mathcal{P}}(G) \geq m$, so $dec(\mathcal{P}) \geq m$. To prove the reverse inequality, note that $\mathcal{P}$ is generated by the set of $\mathcal{P}$-strict, uniquely $\mathcal{P}$-decomposable graphs with minimum decomposability, by Lemma 3.6 and Corollary 3.10. This contains an ordered generating set $\mathcal{G}$, say $G_1 \leq G_2 \leq \cdots$, as constructed in Lemma 4.1. So we have

(a) each G_r is $\mathcal{P}$-strict and uniquely $\mathcal{P}$-decomposable, with $dec_{\mathcal{P}}(G_r) = n$.

Let $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ be a factorisation of $\mathcal{P}$, relative to $\mathcal{G}$, into $n := dec(\mathcal{P})$ indecomposable factors, as constructed in Theorem 4.3. Then we can label the ind-parts of each G_r as $G_{1,r}, \ldots, G_{n,r}$, so that:

(b) for each i, the $G_{i,r}$'s are ordered by inclusion, say $G_{i,1} \leq G_{i,2} \leq \cdots$, and they form a generating set for $\mathcal{P}_i$.

The first part of (b) follows from Lemma 4.2 and the fact that the G_r's are themselves ordered. The second part follows from the proof of Theorem 4.3; in Case 2 it is clear. When $m(F) = k < n$ for some F, let $m(F, G_s) = k$, with F contained in, say, $G_{1,s}, \ldots, G_{k,s}$; then $m(F, G_t) = k$ for all $t \geq s$, and $G_{1,t}, \ldots, G_{k,t}$ are the ind-parts in $\mathcal{Q}_F$. We remove $G_1, \ldots, G_{s-1}$ from the generating set $\mathcal{G}$, and assertion (b) then follows by induction on $dec(\mathcal{P})$.

For each i and j take an arbitrary $X_{i,j} \in \mathcal{P}_i \setminus \mathcal{P}_j$; if $\mathcal{P}_i \setminus \mathcal{P}_j = \emptyset$, then set $X_{i,j}$ to be the null graph K_0. We set $H_i := \bigcup_j X_{i,j}$; note that H_i is in $\mathcal{P}_i$. The important point is that if $\{Y_1, Y_2, \ldots, Y_n\}$ is an unordered $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of some graph G such that, for each $i = 1, 2, \ldots, n$, $H_i \leq G[Y_i]$, then $G[Y_i] \in \mathcal{P}_i$. If not, let $k_1, \ldots, k_r$ be the indices for which $G[Y_{k_j}] \notin \mathcal{P}_{k_j}$; then there is a permutation φ of the k_j's such that $G[Y_{k_j}] \in \mathcal{P}_{\varphi(k_j)} \neq \mathcal{P}_{k_j}$, and we get a contradiction when we consider any $\mathcal{P}_{k_s}$ that is inclusion-wise maximal among the $\mathcal{P}_{k_j}$'s. We must have $H_i \leq G_{i,r}$ for r sufficiently large, so we can omit finitely many G_r's to get:

(c) $H_i \leq G_{i,1}$ for each i.

Properties (a, b, c) guarantee that $(G_{1,r}, \ldots, G_{n,r})$ is the unique *ordered* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of G_r. For each G_r we fix some ordered $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition (it must have at least one such partition). By Lemma 5.1 each $\mathcal{Q}_i$-part is the union of ind-parts of G_r, that is, there is a partition $(S_{1,r}, \ldots, S_{m,r})$ of $\{1, \ldots, n\}$ such that $G_r[\cup_{s \in S_{j,r}} V(G_{s,r})] \in \mathcal{Q}_j$, for each $j = 1, \ldots, m$.

By (b), the partition $(S_{1,r}, \ldots, S_{m,r})$ also works for $G_1, G_2, \ldots, G_{r-1}$. Since there are only finitely many partitions of $\{1, \ldots, n\}$, one of them must appear infinitely often, so we can use this partition of the ind-parts for all r; let it be $(S_1, \ldots, S_m)$.

We want to prove that $\mathcal{Q}_1 = \prod_{s \in S_1} \mathcal{P}_s$. Since $\mathcal{Q}_1$ is irreducible, this will imply that $|S_1| = 1$; the same reasoning applies to $S_j, j = 2, \ldots, m$, so that we must have $m = n$.

Without loss of generality, $S_1 = \{1, \ldots, q\}$. Let A be a graph in $\mathcal{Q}_1$. Note that in G_1, the ind-parts $G_{1,1}, \ldots, G_{q,1}$ form a graph in $\mathcal{Q}_1$. Let v be a vertex of $G_{1,1}$, and let N be the set of neighbours of v in $G_{q+1,1}, \ldots, G_{n,1}$. Let C be the graph formed from $G_1 \cup A$ by adding all possible edges between A and N. C has a $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition with $V(A)$ in the $\mathcal{Q}_1$-part, so it is in $\mathcal{P} = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_m$ and thus has a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition. Now $(G_{1,1}, \ldots, G_{n,1})$ is the unique $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of G_1. If any vertex of A is in $\mathcal{P}_j$, $j > q$, then we could have put

v in $\mathcal{P}_j$, a contradiction, so $A \in \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_q$.

The reverse containment is proved similarly, but requires a bit more work. Let B be a graph in $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_q$, with $(\mathcal{P}_1, \ldots, \mathcal{P}_q)$-partition $(B_1, \ldots, B_q)$. We first create a graph $B' \in \mathcal{P}$ that consists of several copies of B: for every tuple $(i_1, \ldots, i_q)$ such that $\mathcal{P}_1 = \mathcal{P}_{i_1}, \mathcal{P}_2 = \mathcal{P}_{i_2}, \ldots, \mathcal{P}_q = \mathcal{P}_{i_q}$, we put a copy of B with $B_1, \ldots, B_q$ in the $\mathcal{P}_{i_1}, \ldots, \mathcal{P}_{i_q}$ part, respectively. This has an obvious $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition, say $(B'_1, \ldots, B'_n)$; note that B'_j is empty iff $\mathcal{P}_j$ is not equal to any of $\mathcal{P}_1, \ldots, \mathcal{P}_q$.

As before, we take a vertex $v_1 \in G_{1,1}$ and let $N_{\overline{1}}$ be $N(v_1) \setminus V(G_{1,1})$. Similarly, we take $v_2 \in G_{1,2}$ and let $N_{\overline{2}}$ be $N(v_2) \setminus V(G_{1,2}$; and so on for $v_3, \ldots, v_n$ and $N_{\overline{3}}, \ldots, N_{\overline{n}}$. Let D be the graph formed from $G_1 \cup B'$ by adding all possible edges between B'_1 and $N_{\overline{1}}$, B'_2 and $N_{\overline{2}}$, $\ldots$. Then D has a unique $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition, *up to relabeling of identical properties.*

Let D be contained in G_r, for some r. By construction of B', no matter which properties get labeled as $\mathcal{P}_1, \ldots, \mathcal{P}_q$, there will be a copy of B contained in $G_r[V(G_{1,r}) \cup \cdots \cup V(G_{q,r})]$. This subgraph is in $\mathcal{Q}_1$, so we are done. □

We will use the following construction of a generating set for $\mathcal{P}$ to prove unique factorisation. Suppose we are given a factorisation $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$ into indecomposable additive induced-hereditary factors, and, for each i, we are given a generating set $\mathcal{G}_i$ of $\mathcal{P}_i$ and a graph $H_i \in \mathcal{P}_i$. By Lemmas 3.5 and 3.6, the set $\mathcal{G}_i^{\downarrow}[H_i] := \{G \in (\mathcal{G}_i \cap \mathbf{S}(\mathcal{P}_i)) \mid H_i \leq G,\ dec_{\mathcal{P}_i}(G) = 1\}$ is also a generating set for $\mathcal{P}_i$. The $*$-join of these m sets is then a generating set for $\mathcal{P}$, and even if we pick out just those graphs that are strict and have minimum decomposability, we still have a generating set:

$$(\mathcal{G}_1[H_1] * \cdots * \mathcal{G}_m[H_m])^{\downarrow} := \{G' \in \mathbf{S}(\mathcal{P}) \mid dec_{\mathcal{P}}(G') = dec(\mathcal{P}),\ \text{and}\ \forall\, i,\ 1 \leq i \leq m,\ \exists\, G_i \in \mathcal{G}_i^{\downarrow}[H_i],\ G' \in G_1 * \cdots * G_m\}.$$

Theorem 5.3 *An additive induced-hereditary property $\mathcal{P}$ can have only one factorisation with exactly $dec(\mathcal{P})$ indecomposable factors.*

Proof. Let $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_n$ be two factorisations of $\mathcal{P}$ into $n := dec(\mathcal{P})$ indecomposable factors. Label the $\mathcal{P}_i$'s inductively, beginning with $i = n$, so that, for each i, $\mathcal{P}_i$ is inclusion-wise maximal among $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_i$. For each i, j such that $i > j$, if $\mathcal{P}_i \setminus \mathcal{P}_j \neq \emptyset$, then let $X_{i,j} \in \mathcal{P}_i \setminus \mathcal{P}_j$; if $\mathcal{P}_i \setminus \mathcal{P}_j = \emptyset$, then $\mathcal{P}_i = \mathcal{P}_j$ and we set $X_{i,j}$ to be the null graph. For each i, set $H_{i,0} := \bigcup_{j<i} X_{i,j}$. Note $H_{i,0} \in \mathcal{P}_i$. The important point is that if $\{Y_1, Y_2, \ldots, Y_n\}$ is an unordered $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of some graph G such that, for each $i = 1, 2, \ldots, n$, $H_{i,0} \leq G[Y_i]$, then, by reverse induction on i starting at n, $G[Y_i] \in \mathcal{P}_i$.

For each i, let $\mathcal{G}_i = \{G_{i,0}, G_{i,1}, G_{i,2}, \ldots\}$ be a generating set for $\mathcal{P}_i$. We will construct another generating set for each $\mathcal{P}_i$ that will turn out to be contained in some $\mathcal{Q}_j$; for graphs $G_{i,s}, H_{i,s}$, we will use the second subscript to denote which step of our construction we are in.

For each $s \geq 0$, choose a graph $H'_{s+1} \in (\mathcal{G}_1[H_{1,s}, G_{1,s}] * \cdots * \mathcal{G}_n[H_{n,s}, G_{n,s}])^{\downarrow}$, and find an induced supergraph H_{s+1} whose unique $\mathcal{P}$-decomposition with $dec(\mathcal{P})$ parts uniformly respects the obvious decomposition of H'_{s+1}. We label as $H_{i,s+1}$ the ind-part of H_{s+1} that contains the graph from $\mathcal{G}_i[H_{i,s}, G_{i,s}]$. Then, for each i, $H_{i,0} \leq H_{i,1} \leq H_{i,2} \leq \cdots$

For $\mathcal{G}_i[H_{i,s}, G_{i,s}]$ to be non-empty, we must have $H_{i,s} \in \mathcal{P}_i$. We know that the $H_{i,s+1}$'s give an unordered $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$-partition of H_{s+1}. From the earlier remark, for $i = 1, 2, \ldots, n$, $H_{i,s+1} \in \mathcal{P}_i$.

The ind-parts of H_s also form its unique $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_n\}$-partition. Thus, there is some permutation φ_s of $\{1, 2, \ldots, n\}$ such that, for each i, $H_{i,s} \in \mathcal{Q}_{\varphi_s(i)}$. Since there are only finitely many permutations of $\{1, 2, \ldots, n\}$, there must be some permutation φ that appears infinitely often. Now whenever $\varphi_t = \varphi$, we have $H_{i,1} \leq H_{i,2} \leq \cdots \leq H_{i,t} \in \mathcal{Q}_{\varphi(i)}$ so by induced-heredity, for every $s \leq t$, $H_{i,s}$ is in $\mathcal{Q}_{\varphi(i)}$. Therefore, we can take $\varphi_s = \varphi$, for all s. By re-labelling the $\mathcal{Q}_i$'s, we can assume φ is the identity permutation, so that $H_{i,s} \in \mathcal{Q}_i$ for all i and s.

Now for each i and s, $G_{i,s-1} \leq H_{i,s}$, so that $\mathcal{H}_i := \{H_{i,1}, H_{i,2}, \ldots\}$ is a generating set for $\mathcal{P}_i$. But $\mathcal{H}_i \subseteq \mathcal{Q}_i$, so $\mathcal{P}_i = \langle \mathcal{H}_i \rangle \subseteq \mathcal{Q}_i$.

By the same reasoning there is a permutation τ such that $\mathcal{Q}_i \subseteq \mathcal{P}_{\tau(i)}$. We cannot relabel the $\mathcal{P}_i$'s as well, but if $\tau^k(i) = i$, then we have $\mathcal{P}_i \subseteq \mathcal{Q}_i \subseteq \mathcal{P}_{\tau(i)} \subseteq \mathcal{Q}_{\tau(i)} \subseteq \mathcal{P}_{\tau^2(i)} \subseteq \mathcal{Q}_{\tau^2(i)} \subseteq \cdots \subseteq \mathcal{P}_{\tau^k(i)} = \mathcal{P}_i$, so we must have equality throughout; in particular, $\mathcal{P}_i = \mathcal{Q}_i$ for each i. $\square$

Theorem 5.4 *An additive induced-hereditary property $\mathcal{P}$ has a unique factorisation into irreducible additive induced-hereditary factors, and the number of factors is exactly dec($\mathcal{P}$).* $\square$

6 Unique factorization theorem for systems

In this section we will present a common generalization of graphs, hypergraphs, digraphs and other combinatorial systems. We will use the basic elementary notions of category theory (see [12]) and deal only with concrete categories. A concrete category **C** is a collection of *objects* and *arrows* called *morphisms*. An object in a concrete category **C** is "a set with structure". We will denote the *ground-set* of the object A by $V(A)$. The morphism between two objects is a "structure preserving mapping". Obviously, the morphisms of **C** have to satisfy the axioms of the category theory (see e.g. [12], page 1). The natural examples of concrete categories are: **Set** of sets, **FinSet** of finite sets, **Graph** of graphs, **Grp** of groups, **Poset** of partially ordered sets with structure preserving mappings, called homomorphisms of corresponding structures. In our investigations here we will need to consider *isomorphisms* i.e. structure preserving bijections between the ground-sets of objects only.

A simple finite hypergraph $H = (V, E)$ can be considered as a system of its hyperedges $E = \{e_1, e_2, \ldots, e_m\}$, where edges are finite sets and the set of its vertices $V(H)$ is a superset of the union of hyperedges, i.e. $V \supseteq \bigcup_{i=1}^{m} e_i$. The following definition gives a natural generalization of hypergraphs or "set-systems".

Definition 6.1 *Let* **C** *be a concrete category. A* simple system of objects *of* **C** *is an ordered pair $S = (V, E)$, where $E = \{A_1, A_2, \ldots, A_m\}$ is a finite set of the objects of* **C**, *such that the ground-set $V(A_i)$ of each object $A_i \in E$ is a finite set with at least two elements (i.e. there are no loops) and $V \supseteq \bigcup_{i=1}^{m} V(A_i)$.*

For example, graphs can be viewed as systems of objects of a concrete category of two-element sets with bijections as arrows, digraphs as special systems of objects of the category of posets, etc.

To generalize the proof of Unique factorization for coloured hypergraphs to arbitrary simple systems of objects (or shortly systems) we need to define "isomorphism of systems", "disjoint union of systems" and "induced-subsystems", respectively. We can do this in a natural way:

Let $S_1 = (V_1, E_1)$ and $S_2 = (V_2, E_2)$ be two simple systems of objects of a given concrete category $\mathbf{C}$.

The systems S_1 and S_2 are said to be isomorphic if there are two bijections:

$$\phi : V_1 \longleftrightarrow V_2; \qquad \psi : E_1 \longleftrightarrow E_2,$$

such that if $\psi(A_{1i}) = A_{2j}$ then $\phi/V(A_{1i}) : V(A_{1i}) \longleftrightarrow V(A_{2j})$ is an isomorphism of the objects $A_{1i} \in E_1$ and $A_{2j} \in E_2$ in the category $\mathbf{C}$.

The disjoint union of the systems S_1 and S_2 is the system $S_1 \cup S_2 = (V_1 \cup V_2, E_1 \cup E_2)$, where we assume that $V_1 \cap V_2 = \emptyset$.

A system is said to be connected if it cannot be expressed as a disjoint union of two systems.

The subsystem of S_1 induced by the set $U \subseteq V(S_1)$ is $S_1[U]$, with objects $E(S_1[U]) := \{A_{1i} \in E(S_1) | V(A_{1i}) \subseteq U\}$. S_2 is an induced-subsystem of S_1 if it is isomorphic to $S_1[U]$ for some $U \subseteq V(S_1)$.

Using these definitions we can say, analogously as for hypergraphs, that an additive induced-hereditary property of simple systems of objects of a category $\mathbf{C}$ is any class of systems closed under taking induced-subsystems, disjoint union of systems and isomorphism, respectively. To prove the Unique Factorization Theorem for induced hereditary and additive properties of simple systems of objects of a concrete category $\mathbf{C}$ we can follow the notions and constructions given in the previous Sections with some additional technical details, which we will omit here.

References

[1] C. Berge, *Graphs and Hypergraphs*, Second revised edition, North-Holland Mathematical Library 6 (1976) 1-528 (translated from the French by Edward Minieka).

[2] C. Berge, *Hypergraphs. Combinatorics of finite sets*, North-Holland Mathematical Library 45 (1989) 1-255 (translated from the French).

[3] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, *Survey of hereditary properties of graphs*, Discussiones Mathematicae - Graph Theory **17** (1997) 5–50.

[4] M. Borowiecki and P. Mihók, *Hereditary properties of graphs*, in: V.R. Kulli, ed., Advances in Graph Theory (Vishwa International Publication, Gulbarga, 1991) 42–69.

[5] I. Broere and J. Bucko, *Divisibility in additive hereditary properties and uniquely partitionable graphs*, Tatra Mt. Math. Publ. **18** (1999), 79–87.

[6] I. Broere and J. Bucko, P. Mihók, *Criteria for the existence of uniquely partitionable graphs with respect to additive induced-hereditary properties* Discussiones Mathematicae - Graph Theory **22** (2002), 31–37.

[7] A. Farrugia and R.B. Richter, Unique factorisation of additive induced-hereditary properties (to appear in Discussiones Mathematicae - Graph Theory).

[8] D.L. Greenwell, R.L. Hemminger and J. Klerlein, *Forbidden subgraphs* Proc. 4th S-E Conf. Combinatorics, Graph Theory and Computing (Utilitas Math., Winnipeg, Man., 1973) 389–394.

[9] J. Jakubík, *On the lattice of additive hereditary properties of finite graphs*, Discussiones Mathematicae - General Algebra and Applications **22** (2002), 73–86.

[10] P. Mihók, *Unique Factorization Theorem*, Discussiones Mathematicae - Graph Theory **20** (2000), 143–153.

[11] P. Mihók, G. Semanišin and R. Vasky, *Additive and hereditary properties of graphs are uniquely factorizable into irreducible factors*, J. Graph Theory **33** (2000), 44–53.

[12] B.C. Pierce, *Basic Category Theory for Computer Scientists*, Foundations of Computing Series, The MIT Press, Cambridge, Massachusetts 1991.

[13] E.R. Scheinerman, *On the structure of hereditary classes of graphs*, J. Graph Theory **10** (1986) 545–551.